

Cooperative \mathcal{H}_∞ –Estimation for Large-Scale Interconnected Linear Systems

Jingbo Wu*

Valery Ugrinovskii

Frank Allgöwer

Abstract—In this paper, a synthesis method for distributed estimation is presented, which is suitable for dealing with large-scale interconnected linear systems with disturbance. The main feature of the proposed method is that local estimators only estimate a reduced set of state variables and their complexity does not increase with the size of the system. Nevertheless, the local estimators are able to deal with lack of local detectability. Moreover, the estimators guarantee \mathcal{H}_∞ -performance of the estimates with respect to model and measurement disturbances.

I. INTRODUCTION

Estimator design has been an essential part of controller design ever since the development of state-space based controllers. A milestone was laid by the Kalman Filter in 1960 [1]. While in the classical estimator design one estimator is used for the entire system, decentralized and distributed estimators have gained attention since decentralized control became one of the mainstream topics of interest in control theory [2], [3]. More results on decentralized estimation were presented in [4], [5], [6], [7]. In decentralized estimation, typically, a set of estimators are employed to create estimates of local subsystem states with only limited assistance from each other. Couplings between those subsystems are treated as undesirable disturbances. An important requirement of this approach therefore is that the local subsystems are detectable from local measurements.

On the other hand, distributed filtering techniques like Distributed Kalman Filtering were presented in [8], [9], [10], [11]. In a distributed estimation setup, multiple estimators create an estimate of the system's state, while cooperating with each other. Situations are not uncommon where individual estimators are unable to obtain an estimate of the state on its own and cooperation becomes an essential prerequisite [12], [13]. At the same time, the progress in the area of distributed estimation put forward issues of scalability of estimator networks, i.e., there is an interest in distributed estimation methods where the dimension of the local estimators does not increase with the total size of the system. For instance, this is relevant for multi-agent

systems, where the agents are not able to perform a self-measurement, but only receive relative information [14], [15]. Direct applications of the existing distributed estimation algorithms such as those reported in [8], [9], [12], [13] result in the estimators reproducing the entire state of the complete network, and therefore, the order of the estimators grows with the size of the network. However, for cooperative control of multi-agent systems, only local information is required for each agent, making the estimate of the whole network superfluous. This makes the direct application of the mentioned distributed estimation algorithms ineffective with respect to the necessary computation power.

In this paper, our main contribution is the development of a new framework that combines the benefits of both decentralized and distributed estimation. We develop an estimation setup, where local estimators only reproduce a desired subset of state variables and their complexity does not grow with the total size of the system, in contrast to the existing methods for distributed estimation mentioned above. Moreover, cooperation between the local estimators will be used to deal with possible lack of local detectability, which is a major difference to existing results [4], [7]. Therefore, the proposed setup is referred to as *Cooperative Estimation*. In particular, we present an \mathcal{H}_∞ -based design, which in addition provides guaranteed performance with respect to model and measurement disturbances.

This paper significantly extends our results in [16] by allowing a more general class of systems. In [16], we consider dynamically decoupled agents and take relative output measurements. Here, we expand the methodology to general large-scale linear systems, where subsystems may be physically interconnected. This class of systems is of high relevance due to its wide range of applications, such as flexible structures [17] and electrical power grids [18].

The paper is organized as follows: In Section 2, we present some mathematical preliminaries and the system class under consideration. Section 3 presents the methodology of cooperative estimation design with guaranteed \mathcal{H}_∞ -performance, where the first step is an algorithm for a judicious partition of the state space. In order to keep the presentation simple and focused, we make the simplifying assumption that communications between the estimators are perfect. While perturbed communication channels may better reflect real-life applications, this case can indeed also be considered within this estimation setup although the resulting LMI conditions become more cumbersome; cf. [19]. Section 4 illustrates our results with a simulation example and Section 5 concludes the paper, and gives an overview on future work.

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*Corresponding author. J. Wu and F. Allgöwer are with the Institute for Systems Theory and Automatic Control, University of Stuttgart, Stuttgart, Germany. Email: {jingbo.wu, allgower}@ist.uni-stuttgart.de. V. Ugrinovskii is with the School of Information Technology and Electrical Engineering, University of New South Wales at the Australian Defence Force Academy, Canberra, Australia, Email: v.ougrinovski@adfa.edu.au.

II. PRELIMINARIES

Throughout the paper the following notation is used: Let A be a quadratic matrix. If A is positive definite, it is denoted $A > 0$, and we write $A < 0$, if A is negative definite. 0 denotes a matrix of suitable dimension, with all entries equal 0.

A. Communication graphs

In this section we summarize some notation from the graph theory. We use directed, unweighted graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ to describe the communication topology between the individual agents. $\mathcal{V} = \{v_1, \dots, v_N\}$ is the set of vertices, where $v_k \in \mathcal{V}$ represents the k -th agent. $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges, which model the information flow, i.e. the k -th agent receives information from agent j if and only if $(v_j, v_k) \in \mathcal{E}$. The set of vertices that agent k receives information from is called the neighborhood of agent k , which is denoted by $\mathcal{N}^{(k)} = \{j : (v_j, v_k) \in \mathcal{E}\}$. The outdegree of a vertex k is defined as the number of edges in \mathcal{E} , which have v_k as their tail.

B. System model

We consider a large-scale linear time-invariant system, which consists of N interconnected subsystems that are each described by the differential equation

$$\dot{x}_k = A_k x_k + \sum_{j=1}^N A_{kj} x_j + B_k v, \quad (1)$$

$$y_k = C_k x_k + \sum_{j=1}^N C_{kj} x_j + \eta_k, \quad (2)$$

for $k = 1, \dots, N$, where $x_k \in \mathbb{R}^{n_k}$ is the state variable, $y_k \in \mathbb{R}^{r_k}$ is the output, and $v(t), \eta_k(t) \in \mathcal{L}_2[0, \infty)$ are \mathcal{L}_2 -integrable disturbance inputs of subsystem k . The scalar components of x_k will be denoted $x_{k,i}$. Note that in this system, all subsystems are affected by the common disturbance v . This assumption does not lead to loss of generality, since it also captures the case where the subsystems are affected by different disturbances v_k , by simply stacking v_k into one vector.

The global interconnected system can be written as

$$\dot{x} = Ax + Bv, \quad y = Cx + \eta \quad (3)$$

with

$$A = \begin{bmatrix} A_1 & A_{12} & \dots & A_{1N} \\ A_{21} & A_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ A_{N1} & \dots & \dots & A_N \end{bmatrix} \quad C = \begin{bmatrix} C_1 & C_{12} & \dots & C_{1N} \\ C_{21} & C_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ C_{N1} & \dots & \dots & C_N \end{bmatrix}$$

$$B^\top = [B_1^\top \quad \dots \quad B_N^\top]$$

by using the stacked state and disturbance vectors $x = [x_1^\top, \dots, x_N^\top]^\top \in \mathbb{R}^n$ and $\eta = [\eta_1^\top, \dots, \eta_N^\top]^\top$.

Assumption 1: The global plant (A, C) is observable.

It is well known that Assumption 1 is a sufficient condition in the centralized case. In this paper, this assumption is to setup a basic framework under which the state estimation problem under consideration is meaningful.

C. Re-partitioning of the system

In the next section, we will re-partition the vector x for designing local estimators. Associated with the collection of outputs (2), for every $k = 1, \dots, N$, we choose a σ_k -dimensional partial state vector

$$x^{(k)} = \begin{bmatrix} x_{\xi_k(1)} \\ \vdots \\ x_{\xi_k(\sigma_k)} \end{bmatrix}, \quad (4)$$

where $\xi_k(\cdot)$ is a selection function that determines, which scalar components $x_{j,i}$ are included in $x^{(k)}$. This represents a degree of freedom in the design of the estimators and all elements of the global state vector x may be chosen that are relevant to subsystem k . For instance, $x^{(k)}$ may contain x_k , but it does not have to include all of them, if for subsystem k , some parts of its own state are not important. In particular, it is required that all $x_{j,i}$ which contribute towards y_k are included in $x^{(k)}$. As a result, every output y_k can be equivalently expressed as

$$y_k = C^{(k)} x^{(k)} + \eta_k. \quad (5)$$

One possible choice of $x^{(k)}$ is the stacked vector including x_k and all x_j with $C_{kj} \neq 0$. In that case,

$$A^{(k)} = \begin{bmatrix} A_k & A_{k j_1} & \dots \\ A_{j_1 k} & A_{j_1} & \\ \vdots & & \ddots \end{bmatrix}$$

and the rest of the coefficients in (8) are defined in a similar fashion.

The selection function ξ_k is a discrete injective map

$$\xi_k : \{1, \dots, \sigma_k\} \rightarrow \mathcal{Y}, \quad \sigma_k \leq n, \quad (6)$$

where the set $\mathcal{Y} \triangleq \{(k, i) | k = 1, \dots, N; i = 1, \dots, n_k\}$ is defined as the combination of all appearing indexes of the subsystem states and their scalar components $x_{k,i}$. For the ease of notation, we refer to the elements of the set \mathcal{Y} as λ , i.e., $\lambda = (k, i) \in \mathcal{Y}$.

The image of ξ_k is denoted as $I^{(k)}$, $I^{(k)} \subset \mathcal{Y}$, and the inverse map ξ_k^{-1} is an enumeration of the elements of $I^{(k)}$,

$$\xi_k^{-1} : I^{(k)} \rightarrow \{1, \dots, \sigma_k\}, \quad (7)$$

which assigns a position in $x^{(k)}$ to selected components x_λ of the global state vector x .

In general, partial state vectors $x^{(k)}$ may overlap, e.g. $x^{(1)}$ and $x^{(2)}$ may contain a common component x_λ .

For all $k = 1, \dots, N$ the global interconnected system (3) can now be written as

$$\begin{bmatrix} \dot{x}^{(k)} \\ \dot{x}_c^{(k)} \end{bmatrix} = \begin{bmatrix} A^{(k)} & \tilde{A}^{(k)} \\ \tilde{A}_c^{(k)} & A_c^{(k)} \end{bmatrix} \begin{bmatrix} x^{(k)} \\ x_c^{(k)} \end{bmatrix} + \begin{bmatrix} B^{(k)} \\ B_c^{(k)} \end{bmatrix} v \quad (8)$$

by permutation of the states.

For every k , the composition of the matrices $A^{(k)}$, $B^{(k)}$, etc., is determined by the composition of the partial state variable $x^{(k)}$; in turn, the latter is determined by the components of the global state x which estimator k seeks to obtain (see Section III).

III. COOPERATIVE ESTIMATION PROBLEM AND THE ESTIMATOR DESIGN

The problem considered in this paper is to design a local estimator for every subsystem k that creates an estimate for the local partial state variable $x^{(k)}$ using the local measurements y_k described in (5). The vector of local estimates will be denoted

$$\hat{x}^{(k)} = \begin{bmatrix} \hat{x}_{\xi_k(1)}^{(k)} \\ \vdots \\ \hat{x}_{\xi_k(\sigma_k)}^{(k)} \end{bmatrix} \in \mathbb{R}^{\sigma_k},$$

where $\hat{x}_{\lambda}^{(k)}$ is the estimate for x_{λ} computed at subsystem k .

The local estimation error vector is defined as

$$\varepsilon^{(k)} = x^{(k)} - \hat{x}^{(k)} = \begin{bmatrix} x_{\xi_k(1)} - \hat{x}_{\xi_k(1)} \\ \vdots \\ x_{\xi_k(\sigma_k)} - \hat{x}_{\xi_k(\sigma_k)} \end{bmatrix}.$$

We now formally pose the estimator synthesis problem.

Problem 1: Determine a collection of estimates $\hat{x}^{(k)}(t)$, $k = 1, \dots, N$, such that the following two properties are satisfied simultaneously.

- (i) In the absence of model and measurement disturbances (i.e., when $v = 0$, $\eta = 0$), the estimation errors decay so that $\varepsilon^{(k)} \rightarrow 0$ exponentially for all $k = 1, \dots, N$.
- (ii) The estimators (11) provide guaranteed \mathcal{H}_{∞} performance in the sense that

$$\begin{aligned} & \sum_{k=1}^N \int_0^{\infty} \varepsilon^{(k)\top} W^{(k)} \varepsilon^{(k)} dt \\ & \leq \sum_{k=1}^N \int_0^{\infty} (\omega^2 \|v\|^2 + \gamma^2 \|\eta_k\|^2) dt + I_0, \end{aligned} \quad (9)$$

for a positive semi-definite weighting matrix $W^{(k)}$. In (9), $I_0 = \sum_{k=1}^N x_0^{(k)\top} P^{(k)} x_0^{(k)}$ is the cost due to the observer's uncertainty about the initial conditions of the subsystems.

This will be achieved by allowing certain agents to communicate with each other.

A. Communication Requirements

There are two factors, which influence the required communication for the cooperative estimation setup: The first one is detectability of $(A^{(k)}, C^{(k)})$. In the special case of $\tilde{A}^{(k)} = 0$, and $(A^{(k)}, C^{(k)})$ being detectable for all $k = 1, \dots, N$, no communication is necessary at all, as for every subsystem, an estimator can be designed separately. However, these assumptions may not hold in a general case. In particular, in this paper, we do not require that $(A^{(k)}, C^{(k)})$ are detectable for all $k = 1, \dots, N$, which is a major difference compared to existing methods in literature, for instance, [4] and [7]. In fact, even all $(A^{(k)}, C^{(k)})$ may be undetectable.

The second factor which influences the required communication is sparsity of $\tilde{A}^{(k)}$. Ideally, when the partial state $x^{(k)}$ is decoupled from the rest of the system, i.e. $\tilde{A}^{(k)} = 0$, a standard \mathcal{H}_{∞} filter can be employed to carry out the

estimation of $x^{(k)}$ from y_k . However, if $x^{(k)}$ includes a state x_{λ} , which is connected to a state x_{λ^*} that is not a component of $x^{(k)}$, then the problem becomes more challenging. When the connections strength is limited, this can be handled using methods like the Small Gain Theorem. Otherwise, communication may be used to compensate for coupling between $x^{(k)}$ and x_{λ^*} . In this paper, we investigate the latter method.

In order to define the required communication channels, we use an assignment function

$$\zeta : \mathcal{Y} \rightarrow \{0, 1, \dots, N\}, \quad (10)$$

with the property that

$$\lambda \in I^{(\zeta(\lambda))},$$

if $\zeta(\lambda) \neq 0$. Moreover, $\zeta(\lambda) = 0$ only if $\lambda \notin I^{(k)}$ for all $k = 1, \dots, N$. The map $\zeta(\cdot)$ assigns responsibilities in estimating the system's states to the subsystems and their local estimators. In general, $\zeta(\lambda)$ is not unique and there is a degree of freedom in selection of the assignment function. However in the case when $x^{(k)}$'s do not overlap, the assignment function $\zeta(\lambda)$ is unique.

With the definition of the assignment function ζ , we can introduce the assumption on the communication graph used in this paper:

Assumption 2: If a component x_{λ} of $x^{(k)}$ is physically coupled to a state x_{λ^*} , where $\lambda^* \in \mathcal{Y} \setminus I^{(k)}$, then subsystem $j = \zeta(\lambda^*) \neq 0$ can communicate to subsystem k , i.e. $(j, k) \in \mathcal{E}$.

We denote with $I_c^{(k)}$ the set of all indexes $\lambda^* \in \mathcal{Y} \setminus I^{(k)}$ with the property that for all $x_{\lambda^*} \in I_c^{(k)}$, there exists a component x_{λ} of $x^{(k)}$, which is coupled to x_{λ^*} . Assumption 2 reflects the point made above, as the more entries $\tilde{A}^{(k)}$ has, the more communication between the subsystems is required. Some remarks on the realization of this assumption are in order:

Remark 1: In the literature (see e.g. [4], [17]), it is often assumed that the communication topology simply mimics the interconnection topology. Assumption 2 is more general in this respect as the selection of the assignment function ζ and the definition of the partition $x^{(k)}$ imply a certain degree of freedom. In particular, if the partial state $x^{(k)}$ is coupled to some state x_{λ^*} which is not a component of $x^{(k)}$, there are two ways for subsystem k to obtain the required information:

- 1) One intuitive option for subsystem k is to receive an estimate of x_{λ^*} through communication with another subsystem. The transmitting subsystem may be any subsystem, which is assigned to estimate x_{λ^*} by the choice of the assignment function ζ . The function ζ should therefore be selected so that subsystem $j = \zeta(\lambda^*)$ can communicate to subsystem k .
- 2) An alternative option is to include x_{λ^*} into the partial state $x^{(k)}$, and therefore, assign the task of estimating x_{λ^*} to subsystem k . This circumvents the communication requirement of Assumption 2. This method is meaningful, when the local estimator has enough computational power to handle additional coordinates.

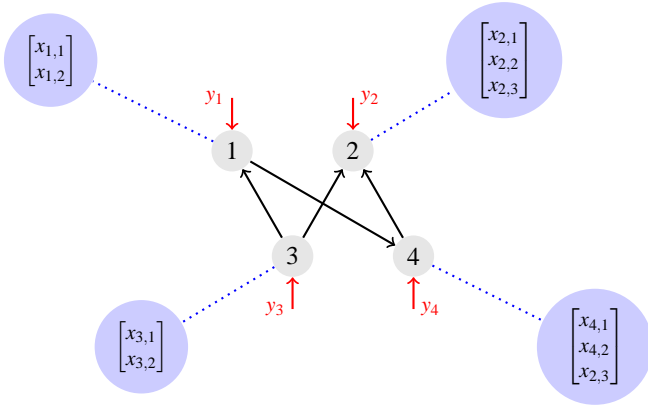


Fig. 1. Example of an admissible estimator structure. The nodes in the center represent the subsystems, which receive measurement information and estimate the partial states $x^{(k)}$ that are written in the outer circles. The black arrows represent the communication links between the subsystems.

Remark 2: In some applications, the communication topology is not a fixed system property, but is a design parameter. In that case, Assumption 2 can be easily realized.

Lemma 1: For all $\lambda \in \mathcal{Y}$ there exists a $k \in \{1, \dots, N\}$, such that $\lambda \in I^{(k)}$.

Proof: Suppose there exists a $\lambda \in \mathcal{Y}$, such that $\lambda \notin I^{(k)}$ for all $k = 1, \dots, N$. By the definition of the partial states $x^{(k)}$ and the selection function ξ_k , the column of C which corresponds to x_λ is 0. Moreover, by the definition of ζ , we have $\zeta(\lambda) = 0$ and thus, it follows from Assumption 2 that there is no partial state $x^{(k)}$ that is coupled to x_λ . Therefore, x_λ is not observable, which contradicts Assumption 1.

An example for the interconnection structure which satisfies Assumption 2 is shown in Figure 1. This communication structure pertains to the numerical example presented later in Section IV. As noted for the partition (4), the vectors $x^{(k)}$ may overlap. Therefore, including a consensus term whenever overlapping estimators can communicate is able to enhance estimation performance of the subsystems and in some cases even facilitates feasibility of the design conditions which will be introduced later in the paper.

B. Formal definition of estimators

The estimator dynamics are now proposed for each subsystem as

$$\begin{aligned} \hat{x}^{(k)} = & A^{(k)} \hat{x}^{(k)} + L^{(k)} (y_k - C^{(k)} \hat{x}^{(k)}) + \sum_{\lambda \in I_c^{(k)}} [\tilde{A}^{(k)}]_\lambda \hat{x}_\lambda^{\zeta(\lambda)} \\ & + K^{(k)} \sum_{j \in \mathcal{N}_k} \left(\sum_{\lambda \in I^{(k)} \cap I^{(j)}} e_{\xi^{-1}(\lambda)} (\hat{x}_\lambda^{(j)} - \hat{x}_\lambda^{(k)}) \right) \end{aligned} \quad (11)$$

with initial condition $\hat{x}_0^{(k)} = 0$, where $[\tilde{A}^{(k)}]_\lambda$ is the column of $\tilde{A}^{(k)}$ which corresponds to x_λ and the unit vector $e_{\xi^{-1}(\lambda)}$ injects the difference $\hat{x}_\lambda^{(j)} - \hat{x}_\lambda^{(k)}$ to the σ_k -dimensional space.

Problem 1 can now be particularized using the estimators (11).

Problem 1': Determine estimator gains $L^{(k)}$, $K^{(k)}$ in (11) such that properties (i) and (ii) of Problem 1 hold.

In order to solve this problem, we define the extended graph $\tilde{\mathcal{G}}$, which will be used in the analysis of the interconnection structure between the subsystems. Let every subsystem be represented by a cluster of σ_k nodes, where vertex $v_{k\lambda}$ represents the estimator state $\hat{x}_\lambda^{(k)}$. The edges of $\tilde{\mathcal{G}}$ are now determined by Algorithm 1. An example for the extended graph is shown in Figure 2, which applies to the numerical example presented in Section IV.

Algorithm 1

Result: Graph $\tilde{\mathcal{G}}$

Set the edge set $\tilde{\mathcal{E}} = \{\}$.

for $k = 1, \dots, N$ **do**

for $\lambda^* \in I_c^{(k)}$ (**interconnection part**) **do**

 Add the edge $(v_{\zeta(\lambda^*)_{\lambda^*}}, v_{k\lambda^*})$ to $\tilde{\mathcal{E}}$, where x_{λ^*} is the component of $x^{(k)}$ that is coupled to x_{λ^*} .

end

for $j \in \mathcal{N}_k$ and $\lambda \in I^{(k)} \cap I^{(j)}$ (**fusion part**) **do**

 Add an edge $(v_{j\lambda}, v_{k\lambda})$ to $\tilde{\mathcal{E}}$.

end

end

The graph generated by Algorithm 1 graphically displays the detailed connection structure of the estimation vectors $\hat{x}^{(k)}$. The out-degree of vertex $v_{k\lambda}$ in the extended graph is denoted by $q(k, \lambda)$. This definition will be used to present our main results on the design of the filter gains, which are given in the next section.

C. Filter gains design

We define the matrices

$$\begin{aligned} N^{(k)} = & \sum_{j \in \mathcal{N}_k} \left(\sum_{\lambda \in I^{(k)} \cap I^{(j)}} e_{\xi^{-1}(\lambda)} e_{\xi^{-1}(\lambda)}^\top \right) \\ Q^{(k)} = & P^{(k)} A^{(k)} + A^{(k)\top} P^{(k)} - G^{(k)} C^{(k)} - (G^{(k)} C^{(k)})^\top \\ & - F^{(k)} N^{(k)} - (F^{(k)} N^{(k)})^\top \\ & + \underbrace{\alpha P^{(k)} + \pi_k}_{\Pi_k} \begin{bmatrix} q(k, \xi_k(1)) P_1^{(k)} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & q(k, \xi_k(\sigma_k)) P_{\sigma_k}^{(k)} \end{bmatrix}, \end{aligned}$$

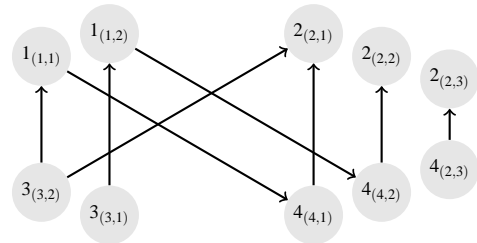


Fig. 2. Example of the extended graph $\tilde{\mathcal{G}}$: The subsystems are now represented by clusters of σ_k vertices, where every vertex represents a single estimator coordinate.

where $G^{(k)} \in \mathbb{R}^{\sigma_k \times r_k}$ and $F^{(k)} \in \mathbb{R}^{\sigma_k \times \sigma_k}$ are unknown matrices, $P^{(k)} \in \mathbb{R}^{\sigma_k \times \sigma_k}$ is a symmetric, positive definite matrix and $P_i^{(k)} \in \mathbb{R}$ is the i -th diagonal element of $P^{(k)}$. π_k and α are positive constants which will later play the role of design parameters. Furthermore, we define $p(\lambda)$ as the diagonal element of $P^{(\zeta(\lambda))}$ which corresponds to x_λ , i.e. the $\xi_{\zeta(\lambda)}^{-1}(\lambda)$ -th diagonal element.

Next, for all $k = 1, \dots, N$, we define the matrices

$$S^{(k)} = \begin{bmatrix} P^{(k)} [\tilde{A}^{(k)}]_{\lambda_1^{(k)}} & P^{(k)} [\tilde{A}^{(k)}]_{\lambda_2^{(k)}} & \dots \end{bmatrix}$$

$$R^{(k)} = \begin{bmatrix} \pi_{\xi(\lambda_1^{(k)})} P(\lambda_1^{(k)}) & 0 & 0 \\ 0 & \pi_{\xi(\lambda_2^{(k)})} P(\lambda_2^{(k)}) & 0 \\ 0 & 0 & \ddots \end{bmatrix},$$

for $\{\lambda_1^{(k)}, \lambda_2^{(k)}, \dots\} = I_c^{(k)}$ and

$$T_j^{(k)} = \begin{bmatrix} F^{(k)} e_{\xi_k^{-1}(\lambda_{1,j})} & F^{(k)} e_{\xi_k^{-1}(\lambda_{2,j})} & \dots \end{bmatrix}$$

$$U_j^{(k)} = \begin{bmatrix} \pi_j P_{\xi_j^{-1}(\lambda_1^{kj})}^{(j)} & 0 & 0 \\ 0 & \pi_j P_{\xi_j^{-1}(\lambda_2^{kj})}^{(j)} & 0 \\ 0 & 0 & \ddots \end{bmatrix},$$

where $\{\lambda_1^{kj}, \lambda_2^{kj}, \dots\} = I^{(k)} \cap I^{(j)}$.

With these definitions, we are ready to present following theorem.

Theorem 1: Consider a group of interconnected LTI systems (1) with local outputs (2). Problem 1 admits a solution in the form of estimators (11) with

$$L^{(k)} = (P^{(k)})^{-1} G^{(k)},$$

$$K^{(k)} = (P^{(k)})^{-1} F^{(k)}, \quad (12)$$

if the matrices $F^{(k)}$, $G^{(k)}$ and $P^{(k)}$, $k = 1, \dots, N$ are a solution of the following LMIs:

$$\left[\begin{array}{ccc|ccc} Q^{(k)} + W^{(k)} & -G^{(k)} & P^{(k)} B^{(k)} & S^{(k)} & T_{j_{1,k}}^{(k)} & \dots & T_{j_{\tau_k,k}}^{(k)} \\ -G^{(k)\top} & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 \\ (P^{(k)} B^{(k)})^\top & 0 & -\omega^2 I & 0 & 0 & 0 & 0 \\ \hline S^{(k)\top} & 0 & 0 & R^{(k)} & 0 & 0 & 0 \\ T_{j_{1,k}}^{(k)\top} & 0 & 0 & 0 & U_{j_{1,k}}^{(k)} & 0 & 0 \\ \vdots & 0 & 0 & 0 & 0 & \ddots & 0 \\ T_{j_{\tau_k,k}}^{(k)\top} & 0 & 0 & 0 & 0 & 0 & U_{j_{\tau_k,k}}^{(k)} \end{array} \right] < 0 \quad (13)$$

with $\{j_{1,k}, j_{2,k}, \dots, j_{\tau_k,k}\} = \mathcal{N}_k$.

Proof: The estimator error dynamics at node k are

$$\dot{\varepsilon}^{(k)} = (A^{(k)} - L^{(k)} C^{(k)}) \varepsilon^{(k)} + \sum_{\lambda \in I_c^{(k)}} [\tilde{A}^{(k)}]_{\lambda} \varepsilon_{\lambda}^{(\zeta(\lambda))}$$

$$+ K^{(k)} \sum_{j \in \mathcal{N}_k} \left(\sum_{\lambda \in I^{(k)} \cap I^{(j)}} e_{\xi^{-1}(\lambda)} (\varepsilon_{\lambda}^{(j)} - \varepsilon_{\lambda}^{(k)}) \right)$$

$$- L^{(k)} \eta^{(k)} + B^{(k)} v.$$

We use a Lyapunov function

$$V(\varepsilon) = \sum_{k=1}^N \underbrace{\varepsilon^{(k)\top} P^{(k)} \varepsilon^{(k)}}_{V^{(k)}(\varepsilon^{(k)})},$$

where $V^{(k)}(\varepsilon^{(k)})$ are the individual components of $V(\varepsilon)$.

The Lie derivative of $V^{(k)}(\varepsilon^{(k)})$ is

$$\dot{V}^{(k)}(\varepsilon^{(k)}) = 2\varepsilon^{(k)\top} P^{(k)} (A^{(k)} - L^{(k)} C^{(k)}) \varepsilon^{(k)}$$

$$+ 2\varepsilon^{(k)\top} P^{(k)} (-L^{(k)} \eta^{(k)} + B^{(k)} v)$$

$$+ 2\varepsilon^{(k)\top} P^{(k)} \sum_{\lambda \in I_c^{(k)}} [\tilde{A}^{(k)}]_{\lambda} \varepsilon_{\lambda}^{(\zeta(\lambda))}$$

$$+ 2\varepsilon^{(k)\top} P^{(k)} K^{(k)} \sum_{j \in \mathcal{N}_k} \sum_{\lambda \in I^{(k)} \cap I^{(j)}} e_{\xi^{-1}(\lambda)} (\varepsilon_{\lambda}^{(j)} - \varepsilon_{\lambda}^{(k)})$$

$$= 2\varepsilon^{(k)\top} P^{(k)} (A^{(k)} - L^{(k)} C^{(k)} - K^{(k)} N^{(k)}) \varepsilon^{(k)}$$

$$+ 2\varepsilon^{(k)\top} P^{(k)} (-L^{(k)} \eta^{(k)} + B^{(k)} v)$$

$$+ 2\varepsilon^{(k)\top} P^{(k)} \sum_{\lambda \in I_c^{(k)}} [\tilde{A}^{(k)}]_{\lambda} \varepsilon_{\lambda}^{(\zeta(\lambda))}$$

$$+ 2\varepsilon^{(k)\top} P^{(k)} K^{(k)} \sum_{j \in \mathcal{N}_k} \sum_{\lambda \in I^{(k)} \cap I^{(j)}} e_{\xi^{-1}(\lambda)} \varepsilon_{\lambda}^{(j)}$$

With the filter gains (12) and the LMIs (13) it can be obtained that

$$\dot{V}^{(k)}(\varepsilon) = \varepsilon^{(k)\top} (Q^{(k)} - \alpha P^{(k)} - \Pi_k) \varepsilon^{(k)}$$

$$- 2\varepsilon^{(k)\top} G^{(k)} \eta^{(k)} + 2e^{(k)\top} P^{(k)} B^{(k)} v$$

$$+ 2\varepsilon^{(k)\top} P^{(k)} \sum_{\lambda \in I_c^{(k)}} [\tilde{A}^{(k)}]_{\lambda} \varepsilon_{\lambda}^{(\zeta(\lambda))}$$

$$+ 2\varepsilon^{(k)\top} F^{(k)} \sum_{j \in \mathcal{N}_k} \sum_{\lambda \in I^{(k)} \cap I^{(j)}} e_{\xi^{-1}(\lambda)} \varepsilon_{\lambda}^{(j)}$$

$$\leq \sum_{\lambda \in I_c^{(k)}} \varepsilon_{\lambda}^{(\zeta(\lambda))\top} \pi_{\zeta(\lambda)} p(\lambda) \varepsilon_{\lambda}^{(\zeta(\lambda))}$$

$$+ \sum_{j \in \mathcal{N}_k} \sum_{\lambda \in I^{(k)} \cap I^{(j)}} \varepsilon_{\lambda}^{(j)\top} \pi_j P_{\xi_j^{-1}(\lambda)}^{(j)} \varepsilon_{\lambda}^{(j)}$$

$$- \varepsilon^{(k)\top} W^{(k)} \varepsilon^{(k)} + \gamma^2 \eta^{(k)\top} \eta^{(k)} + \omega^2 v^\top v$$

$$- \alpha \varepsilon^{(k)\top} P^{(k)} \varepsilon^{(k)} - \varepsilon^{(k)\top} \Pi_k \varepsilon^{(k)}.$$

Summing up the $V^{(k)}$ s, it holds for V that

$$\dot{V}(\varepsilon) \leq \sum_{k=1}^N \pi_k \sum_{\lambda \in I^{(k)}} q(k, \lambda) \varepsilon_{\lambda}^{(k)\top} P_{\lambda}^{(k)} \varepsilon_{\lambda}^{(k)}$$

$$- \sum_{k=1}^N \varepsilon^{(k)\top} W^{(k)} \varepsilon^{(k)} + \sum_{k=1}^N \gamma^2 \eta^{(k)\top} \eta^{(k)} + \sum_{k=1}^N \omega^2 v^\top v$$

$$- \sum_{k=1}^N \alpha \varepsilon^{(k)\top} P^{(k)} \varepsilon^{(k)} - \sum_{k=1}^N \varepsilon^{(k)\top} \Pi_k \varepsilon^{(k)}$$

$$\dot{V}(\varepsilon) \leq -\alpha \sum_{j=1}^N \underbrace{\varepsilon^{(k)\top} P^{(k)} \varepsilon^{(k)}}_{V^{(k)}} - \sum_{k=1}^N \varepsilon^{(k)\top} W^{(k)} \varepsilon^{(k)}$$

$$+ \sum_{k=1}^N \gamma^2 \eta^{(k)\top} \eta^{(k)} + \sum_{k=1}^N \omega^2 v^\top v \quad (14)$$

Integrating both sides of (14) on the interval $[0, T]$, we obtain

$$\begin{aligned} V(\varepsilon(T)) + \sum_{k=1}^N \int_0^T \varepsilon^{(k)\top} W^{(k)} \varepsilon^{(k)} dt \\ \leq \sum_{k=1}^N \int_0^T \left(\omega^2 \|v\|^2 + \gamma^2 \|\eta^{(k)}\|^2 \right) dt + \sum_{k=1}^N \varepsilon_0^{(k)\top} P^{(k)} \varepsilon_0^{(k)}. \end{aligned}$$

As $V(e(T)) \geq 0$ and with the zero initial conditions of the observer states, it follows that

$$\sum_{k=1}^N \int_0^T \varepsilon^{(k)\top} W^{(k)} \varepsilon^{(k)} dt \leq \sum_{k=1}^N \int_0^T \left(\omega^2 \|v\|^2 + \gamma^2 \|\eta^{(k)}\|^2 \right) dt + I_0.$$

Letting $T \rightarrow \infty$, this satisfies Property (ii) of Problem 1.

Moreover, if $\xi_k = 0$ and $\eta_k = 0$ for all $k = 1, \dots, N$, then it follows from (14) that

$$\dot{V}(\varepsilon) \leq -\alpha V,$$

which implies that Property (i) of Problem 1 holds. \blacksquare

Note that the choice of α determines the convergence speed of the estimators, where a larger α enforces faster convergence of the estimates. However, larger values of α typically lead to higher filter gain values.

The salient feature of the resulting cooperative estimators (11) is that these estimators are local and their complexity does not increase with the total size of the network. In this sense, the method presented in this paper is scalable and guarantees \mathcal{H}_∞ -type performance. In contrast, a direct application of the algorithms developed in [8], [9] and [12], [13], to the problem considered here would result in the order of the estimators growing with the size of the network. Some remarks on the solution of the LMIs (13) are in order now.

Remark 3: As it can be seen from the LMIs (13), the solution to design problem presented here involves solving coupled LMIs. When the nature of the application allows for these LMIs to be solved offline, this can be done in a centralized manner. The resulting gain matrices $L^{(k)}$, $K^{(k)}$ can then be deployed to the filters, this will ensure that while the estimation algorithm is running, the estimators are fully distributed. Alternatively, it was shown in [13], that such LMIs can be solved with gradient descent type algorithms that allow distributed implementation.

Remark 4: As noted before, the choice of the partial state vectors (4) is not unique. As a special case, the choice $x^{(k)} = x$ for all $k = 1, \dots, N$ yields local estimators similar to [12], [13].

IV. SIMULATION EXAMPLE

As an illustrative example, we consider a 9-dimensional system, which is composed of 4 subsystems

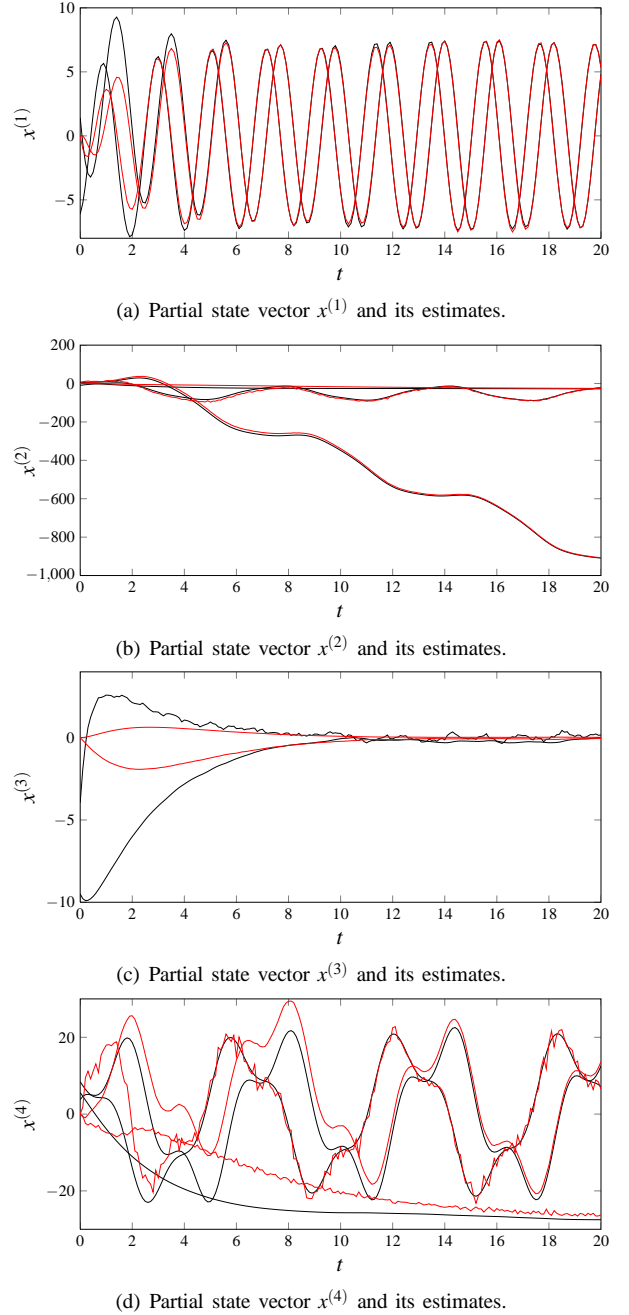


Fig. 3. Plots of the local states. Black lines represent the actual partial state vectors $x^{(k)}$, red lines represent the estimated states $\hat{x}^{(k)}$.

$$\dot{x} = \begin{bmatrix} 0 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -3 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} v \quad (15)$$

with $x = [x_{1,1} \ x_{1,2} \ x_{2,1} \ x_{2,2} \ x_{2,3} \ x_{3,1} \ x_{3,2} \ x_{4,1} \ x_{4,2}]^T$, and four measurements

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 \end{bmatrix} x + \eta, \quad (16)$$

which are associated with the four subsystems. Thus, the partial state vectors (4) can be chosen as shown in Figure 1. Note that although the system has a dimension of 9 and all the coordinates of the state vector are tightly coupled, none of the estimators need to handle more than three coordinates. In particular, $(A^{(2)}, C^{(2)})$ as defined in (8) is not detectable. This emphasizes the benefits of our algorithm, which can deal with situations in which subsystems are individually not detectable.

The graphs representing the interconnection topology are shown in Figure 1 and 2. After applying the estimators (11), we obtain simulation results shown in Figure 3. All local estimators obtain a correct estimation of their respective partial state vector. Moreover, property (ii) of Problem 1 is satisfied with performance values of $\gamma = 11.1$ and $\omega = 8.4$.

V. CONCLUSION

In this paper, we presented a \mathcal{H}_∞ -based approach to cooperative state estimation for linear interconnected large-scale systems, such as multi-agent systems. In order to achieve scalability of the estimation setup, we required the local estimators to estimate local states only. We establish an algorithm for interconnecting the local estimators, whereby both physical couplings and detectability issues can be handled. Moreover, design conditions are presented to guarantee \mathcal{H}_∞ -performance with respect to both model and measurement disturbances.

Further research will include the distributed calculation of the filter gains (12). Moreover, an interesting problem to consider is concerned with the application of the cooperative estimation algorithm to cooperative and decentralized control problems. This may yield interesting results with respect to the \mathcal{H}_∞ -performance that can still be guaranteed.

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